Pair Correlation Function and Nonlinear Kinetic Equation for a Spatially Uniform Polarizable Nonideal Plasma

V. V. Belyi, Yu. A. Kukharenko, and J. Wallenborn

1IZMIRAN, Troitsk, Moscow region, 142092, Russia
2UITFRAN, Moscow, Russia
3Physique Statistique, Plasmas et Optique non Linéaire, Université Libre de Bruxelles, C.P. 231, 1050 Bruxelles, Belgium

(Received 27 October 1995)

Taking into account the first non-Markovian correction to the Balescu-Lenard equation, we have derived an expression for the pair correlation function and a nonlinear kinetic equation valid for a nonideal polarized classical plasma. This last equation allows for the description of the correlational energy evolution and shows the global conservation of energy with dynamical polarization.

PACS numbers: 52.25.Dg, 05.20.Dd

The importance of the polarization effects on plasmas in the kinetic regime has been recognized for a long time. A first attempt to include these effects in the linearized collisional integral was made by Gasirowicz, Neuman, and Riddell [1]. Later, the nonlinear equation for weakly coupled polarizable plasma was derived by Balescu [2] with the help of Prigogine’s diagram techniques [3] and by Lenard [4] who solved the Bogoliubov equation [5] for the pair correlation function in the plasma approximation. These results were generalized to the quantum case by Konstantinov and Perel [6] and by Wyeld and Pines [7].

Kadanoff and Baym [8] and Klimontovich [9] have noticed that the Balescu-Lenard (BL) equation takes into account the polarization of the system only in the collision integral while the thermodynamics corresponds to the ideal gas; the dissipative and nondissipative phenomena are not treated on an equal footing. They have shown that this discrepancy can be avoided if non-Markovian effects are taken into account. This means that Bogoliubov’s condition [5] of the total synchronization of all correlation functions with the one-particle distribution function (df) must be omitted. Klimontovich [10] wrote the system of equations for the one-particle df and the pair correlations for the electric field and for the charge density, but he did not solve the equation for the pair correlation function and thus did not obtain a closed kinetic equation. On the other hand, Résibois [11] and Dorfman and Cohen [12] have formally derived the fully non-Markovian generalization of the BL equation, but their results are not easily tractable in a practical case. In particular, they did not give the explicit expression for the correlation energy.

In this Letter we solve, in the plasma approximation, the equation for the pair correlation function considering the first non-Markovian correction. In this way, we obtain a nonlinear kinetic equation which generalizes the BL equation for weakly nonideal plasma. This equation, which includes the dynamical screening of the interaction potential, describes correctly the conservation of the total energy in a nontrivial way.

The non-Markovian correction, which is responsible for the offset of the variation of kinetic energy by the variation of potential energy, is of order $\gamma^2$, where $\gamma$ is the plasma parameter, while the BL collision integral, which is the Markovian contribution in the plasma approximation, is of order $\gamma$. The BL approximation is thus consistent with the conservation of kinetic energy without potential energy. However, Markovian contributions of order $\gamma^2$, which also conserve kinetic energy, are not included in the plasma approximation. We do not consider these contributions which give rise to a small quantitative correction of the system relaxation time; we focus only on the important qualitative effect related to the potential energy balance.

The importance of nonlocal effects (in space and time) for the description of dense fluids has been recognized since the early works of Akcasu and Duderstadt [13] and Forster and Martin [14] on memory function formalism for the correlations of phase space density fluctuations. However, this formalism corresponds to the linearized kinetic theory near equilibrium.

Far from equilibrium, memory effects are important when the relaxation time becomes of the same order of magnitude as the duration of a collision, that is in the case of dense systems in which the evolution of the potential energy must be considered. Recently, papers on non-Markovian nonlinear kinetic equations were published. They treat the first density correction to the Uehling-Uhlenbeck equation [15] and the case of the quantum Landau equation [16] which accounts for potential energy in lowest order only. Here, we present a kinetic equation that shows the global energy conservation in an approximation which, at equilibrium, is equivalent to the Debye-Hückel one.

We consider a spatially uniform weakly nonideal multicomponent plasma. Out of equilibrium, its evolution...
can be described by a kinetic equation which is derived from the Bogoliubov-Born-Green-Kirwood-Yvon (BBGKY) hierarchy by making the so-called plasma approximation: triple correlation function as well as interaction between correlated particles are neglected. In this way, the set of equations for the one-particle df $f_a(p, t)$ and for the pair correlation function $g_{ab}(p_a, p_b, r_{ab}, t)$ is closed. In Fourier space, they take the form

$$\frac{\partial}{\partial t} f_a(p, t) = J_a(p, t) = - \frac{i}{8\pi^3} \sum_b \int dk \; dp_2 \; \Phi_{ab}(k) \times k \cdot \frac{\partial}{\partial p_1} g_{ab}(p, p_2, k, t),$$

where $J_a(p, t)$ is the collision integral and where $g_{ab}(p, p_2, k, t)$ is the solution of the equation

$$\left[ \frac{\partial}{\partial t} + i k \cdot (v_1 - v_2) \right] g_{ab}(p, p', t) - i k \cdot \frac{\partial}{\partial p} f_a(p, t) \sum_c \Phi_{ac}(k) \int dp_3 \; g_{bc}(p, p_3, t) + i k \cdot \frac{\partial}{\partial p_2} f_a(p, t) \sum_c \Phi_{bc}(k) \int dp_3 \; g_{ac}(p, p_3, t) = i \Phi_{ab}(k) k \cdot \left( \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2} \right) f_a(p, t)f_b(p_2, t).$$

We solve Eq. (4) by using the so-called Green’s function method [17] which should not be confused with the quantum field theory techniques. Taking into account the method [17] which should not be confused with the

$$\left( \partial_t - L \cdot \frac{\partial}{\partial t} \right) R(t, t') = \delta(t - t') L,$$

where $L = L\cdot \frac{\partial}{\partial t}$, where $\mu$ is a parameter which specifies the distribution function time scale, and consider $\bar{\Psi}(\mu t) = \Gamma(\mu t) - \Gamma(0)$ as a perturbation. Equation (7) is then written as

$$\left( \partial_t - L \cdot \frac{\partial}{\partial t} \right) R(t, t') + \bar{\Psi}(\mu t) \cdot R(t, t') = \delta(t - t') L$$

with $L = L\cdot \frac{\partial}{\partial t} - \Gamma(0)$. At order zero, Eq. (8) becomes

$$\left( \partial_t - L \cdot \frac{\partial}{\partial t} \right) R(t, t') = \delta(t - t') L$$

or

$$\left( \partial_t - L \cdot \frac{\partial}{\partial t} \right) \bar{R}(t, t') = \bar{R}_0^{-1}(t - t'),$$

where we have introduced the inverse Green’s function $R_0(t - t')$. Order one of Eq. (8) is

$$\left( \partial_t - L \cdot \frac{\partial}{\partial t} \right) \bar{R}_1(t, t') + \bar{\Psi}(\mu t) \cdot R_0(t, t') = 0,$$

or using Eq. (10)

$$\bar{R}_1(t, t') = - \int_{-\infty}^{\infty} dt_1 \; R_0(t, t_1) \cdot \bar{\Psi}(\mu t_1) \cdot R_0(t_1, t - t').$$

Finally, by a change of integration variable $t_1 \rightarrow t_1 + t'$, we obtain

$$\bar{R}_1(t, t') = - \int_{-\infty}^{\infty} dt_1 \; R_0(t - t_1) \cdot \bar{\Psi}(\mu t_1 + \mu t') \cdot R_0(t_1) = R_1(t - t', \mu t').$$

The procedure is easily iterated to any order in perturbation. We thus have

$$\bar{R}(t, t') = R(t - t', \mu t'),$$

and Eq. (6) becomes

$$\left( \partial_t - L \cdot \frac{\partial}{\partial t} \right) \bar{R}(t, t') = \delta(t - t') L.$$
The Markovian limit of Eq. (15) is clearly obtained when \( t \) is large enough and when \( \mathcal{R}(\tau, \mu(t - \tau)) \) is such that we can made \( t - \tau \approx t \) in the integral.

In the present Letter, we consider only this Markovian limit and the first non-Markovian correction in Eq. (15). Let us point out that it is not a trivial approximation since it leads to the total energy conservation. We proceed by expanding the rhs of Eq. (15):

\[
g(t) = \int_0^\infty d\tau [1 - \tau \partial_\tau] R(\tau, \mu(t - \tau)) \cdot A(\mu(t)).
\]

By introducing the resolvent \( \mathcal{R}(z, \mu t) \) as the Laplace transform of \( R(\tau, \mu t) \),

\[
\mathcal{R}(\tau, \mu t) = \int_0^\infty d\tau \exp(i\zeta\tau) R(\tau, \mu t), \quad \text{Im} \ z > 0,
\]

Equation (16) takes the form (since from now on all quantities depend only on the df time scale we put \( \mu = 1 \)):

\[
g(t) = \lim_{\zeta \to i\infty} \left( 1 + i\zeta \frac{\partial}{\partial \zeta} \right) \mathcal{R}(z, t) \cdot A(t).
\]

Finally, using the identity \( \frac{\partial}{\partial \zeta} \mathcal{R}(z, t) = -\mathcal{R}(z, t) \cdot R(z, t) \), we obtain

\[
g(t) = \lim_{\zeta \to i\infty} \left[ \mathcal{R}(z, t) \cdot A(t) - i\zeta \mathcal{R}(z, t) \cdot R(z, t) \cdot A(t) \right].
\]

The solution of Eq. (7) with \( \Gamma(t) \) constant in time is well known. Green’s function \( R_0(t - t') \) is given as a product of two propagators which are solutions of the linear Vlasov equation (see, e.g., Refs. [18,19]). The remaining calculations are straightforward, although tedious.

The results are

\[
\delta_{ab}^1(\mathbf{p}_1, \mathbf{p}_2, k, t) = i \frac{\partial}{\partial t} \left[ i\pi \delta^\prime_-(\mathbf{k} \cdot \mathbf{v}_1 - \mathbf{k} \cdot \mathbf{v}_2) G_{ab}(\mathbf{p}_1, \mathbf{p}_2, k, t) + \sum_c \Phi_{bc}(k) \frac{\partial}{\partial \mathbf{p}_1} \delta^\prime_-(\mathbf{k} \cdot \mathbf{v}_1 - \mathbf{k} \cdot \mathbf{v}_2) \right]
\]

\[
\times f_b(\mathbf{p}_2, t) \int d\mathbf{p}_3 i\pi \delta^\prime_-(\mathbf{k} \cdot \mathbf{v}_3 - \mathbf{k} \cdot \mathbf{v}_2) G_{ac}(\mathbf{p}_1, \mathbf{p}_3, k, t) - \sum_d \Phi_{cd}(\mathbf{k}) \frac{\partial}{\partial \mathbf{p}_2} \delta^\prime_-(\mathbf{k} \cdot \mathbf{v}_1 - \mathbf{k} \cdot \mathbf{v}_2)
\]

\[
\times \mathbf{k} \cdot \frac{\partial}{\partial \mathbf{p}_1} f_a(\mathbf{p}_1, t) \int d\mathbf{p}_3 i\pi \delta^\prime_-(\mathbf{k} \cdot \mathbf{v}_3 - \mathbf{k} \cdot \mathbf{v}_2) G_{ac}(\mathbf{p}_2, \mathbf{p}_3, k, t) + \sum_{c,d} i\pi \delta^\prime_-(\mathbf{k} \cdot \mathbf{v}_1 - \mathbf{k} \cdot \mathbf{v}_2) \Phi_{ad}(\mathbf{k})
\]

\[
\times \mathbf{k} \cdot \frac{\partial}{\partial \mathbf{p}_1} f_a(\mathbf{p}_1, t) \int d\mathbf{p}_3 \left[ \frac{\delta_-(\mathbf{k} \cdot \mathbf{v}_1 - \mathbf{k} \cdot \mathbf{v}_2)}{|\mathbf{v}_1| \cdot |\mathbf{k} \cdot \mathbf{v}_1|} + \frac{\delta_-(\mathbf{k} \cdot \mathbf{v}_1 - \mathbf{k} \cdot \mathbf{v}_2)}{|\mathbf{v}_1| \cdot |\mathbf{k} \cdot \mathbf{v}_1|} \right] \mathbf{k} \cdot \frac{\partial}{\partial \mathbf{p}_2} f_b(\mathbf{p}_2, t) \int d\mathbf{p}_4 i\pi \delta^\prime_-(\mathbf{k} \cdot \mathbf{v}_3 - \mathbf{k} \cdot \mathbf{v}_2) G_{cd}(\mathbf{p}_3, \mathbf{p}_4, k, t),
\]

where \( \delta^\prime_-(\mathbf{x}) \) is the derivative of \( \delta_-(\mathbf{x}) \). It is interesting to remark that the last three terms of Eq. (24) have a structure similar to Eq. (21) [with Eq. (22)] in which one-
The correction to the collision integral is obtained from Eq. (1):

\[ J_1^d(p_1, t) = -\frac{i}{8\pi^3} \sum_b \int \frac{d\mathbf{k} \, d\mathbf{p}_2 \, \Phi_{ab}(\mathbf{k}) \mathbf{k}}{2m_a} \cdot \frac{\partial}{\partial \mathbf{p}_1} g_{ab}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}, t). \]  

(25)

The kinetic equation (1) including the collision integral (25) can be solved by using Grad’s method, for instance. In the case of a two temperature plasma the non-Markovian contribution results in a correction of the thermodynamic coefficients.

The conservation laws are easily verified with the following symmetry property in mind:

\[ g_{ab}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}, t) = g_{ba}(\mathbf{p}_2, \mathbf{p}_1, -\mathbf{k}, t). \]  

(26)

The particle conservation is a trivial consequence of expression (1) while the momentum conservation follows directly from Eqs. (1) and (26). We know that kinetic energy is conserved by \( J_0^d(p_1, t) \), the Balescu-Lenard collision integral. Therefore, the variation of the kinetic energy is due to the non-Markovian part of the collision integral \( J_1^d(p_1, t) \):

\[ \frac{\partial}{\partial t} \sum_a \left( \frac{p_i^2}{2m_a} \right) = -\frac{1}{8\pi^3} \sum_{ab} \int \frac{d\mathbf{k} \, d\mathbf{p}_1 \, d\mathbf{p}_2 \, \Phi_{ab}(\mathbf{k})}{2m_a} \times (\mathbf{k} \cdot \mathbf{v}_1 - \mathbf{k} \cdot \mathbf{v}_2) g_{ab}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}, t), \]  

(27)

where we used Eqs. (1), (25), and (26). The three last terms of the rhs of Eq. (24) give no contribution to the rhs of Eq. (27) since they are proportional to \( \int d\mathbf{p} \frac{\partial}{\partial \mathbf{p}} f_a(\mathbf{p}, t) = 0 \). The first term leads to

\[ \frac{\partial}{\partial t} \sum_a \left( \frac{p_i^2}{2m_a} \right) = -\frac{1}{8\pi^3} \frac{1}{2} \sum_{ab} \int d\mathbf{k} \, d\mathbf{p}_1 \, d\mathbf{p}_2 \times \Phi_{ab}(\mathbf{k}) g_{ab}^0(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}, t), \]  

(28)

The rhs of Eq. (28) is precisely minus the variation of the potential energy at the order of approximation we consider. This potential energy can be written explicitly as

\[ U = \frac{1}{16\pi^3} \sum_{ab} \int d\mathbf{k} \, d\mathbf{p}_1 \, d\mathbf{p}_2 \, \Phi_{ab}(\mathbf{k}) g_{ab}^0(t, \mathbf{k}, \mathbf{p}_1, \mathbf{p}_2), \]

\[ = \frac{1}{16\pi^3} \sum_b \int d\mathbf{k} \, d\mathbf{p}_2 \, \Phi_{bb}(\mathbf{k}) f_b(\mathbf{p}_2, t) \left[ \frac{1 - |\mathbf{v} \cdot \mathbf{k}|^2}{1 - |\mathbf{v} \cdot \mathbf{k}|^2} \right] \]

\[ = \frac{1}{32\pi^3} \sum_b \int d\mathbf{k} \, d\mathbf{p}_1 \, d\mathbf{p}_2 \, \frac{\Phi_{bb}(\mathbf{k})}{\mathbf{v}_1 \cdot \mathbf{k} - \mathbf{v}_2 \cdot \mathbf{k}} \times \left[ \frac{1}{|\mathbf{v}_1 \cdot \mathbf{k}|^2} + \frac{1}{|\mathbf{v}_2 \cdot \mathbf{k}|^2} \right] \times \left[ f_b \mathbf{k} \cdot \frac{\partial}{\partial \mathbf{p}_1} f_a - f_a \mathbf{k} \cdot \frac{\partial}{\partial \mathbf{p}_2} f_b \right]. \]  

(29)

At equilibrium, this expression becomes the Debye-Hückel energy [17–19]. Equation (28) achieves the consistency of the theory.

Let us conclude with two final remarks. First, the expression of \( J_1^d(p_1, t) \) given by Eq. (25) with Eq. (24), which is very useful to verify the conservation laws, can be expressed in a more compact form which can be used with benefit in other applications. Second, in many problems, nonlocality in space must be considered as well as non-Markovian character (or nonlocality in time); we hope to consider this topic in the near future.

We are grateful to Professor R. Balescu for helpful discussion. This work is supported by INTAS under Contract No. 1010–CT93–0015.