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## Chapter 7

# Systematic error of standard STRO when non-modulated radio propagation in the ionosphere is described

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### § 1. Introduction

The modified STRO version not only makes dispersion refraction a mathematically describable wave effect, but also specifies the description of ordinary refraction of non-modulated monochromatic waves.

In this chapter we will consider, how much is the value of the standard STRO error as compared to such an error of the modified version in the case when HF radio propagation in the ionosphere is described.

First, we compare both STRO versions with the classical exact solution of the Klein–Gordon wave equation for refraction in a linear layer  $\omega_L^2$ .

Then, we realize numerical ray tracing, using two methods for certain standard models of the ionosphere, and compare the results.

### § 2. Wave refraction in linear layer $\omega_L^2$

Let us consider the case when a plane homogeneous monochromatic wave with frequency  $\omega$  is incident on an inhomogeneous half-space  $y \geq 0$  from a homogeneous half-space  $y < 0$  at angle  $\alpha$ .

Assume, that the distribution of the medium parameter  $\omega_L^2$  depends only on the vertical coordinate  $y$  and is specified as

$$\omega_L^2(y) = 0, \quad y < 0;$$

$$\omega_L^2(y) = \beta y, \quad y \geq 0.$$

The exact solution for the wave field is specified by KGE (1.7):

$$\nabla^2 U - \frac{1}{c^2} \frac{\partial^2 U}{\partial t^2} - \frac{\omega_L^2}{c^2} U = 0.$$

To trace space-time rays, it is necessary to integrate Eq. (4.8):

$$\frac{d\mathbf{r}}{dt} = \mathbf{V}_g,$$

where the group velocity vector  $\mathbf{V}_g$  can be defined by formula (4.10):

$$\mathbf{V}_g = c^2 \frac{\mathbf{k}}{\omega}.$$

In standard RO, the group velocity derivative is defined as (4.13):

$$\frac{d\mathbf{V}_g}{dt} = -\frac{c^2 \omega_L}{\omega^2} \nabla \omega_L.$$

In our modified STRO, this derivative for monochromatic wave without transverse frequency modulation has an additional corrective term (4.30):

$$\frac{d\mathbf{V}_g}{dt} = -\frac{c^2 \omega_L}{\omega^2} \nabla \omega_L + \frac{c^2 \omega_L^3}{\omega^4} \nabla_{\perp} \omega_L. \quad (7.1)$$

We now obtain the exact solution based on the initial Klein–Gordon wave equation.

We will seek the solution to the equation in an inhomogeneous half-space  $y \geq 0$  in the form

$$U(x, y, t) = A(y) \exp\left\{\left(\frac{\omega}{c} \cos \alpha\right)x - \omega t\right\}. \quad (7.2)$$

Substituting (7.2) in KGE (1.7), for the  $A(y)$  function we obtain

$$\frac{\partial^2 A}{\partial y^2} - \left\{ \frac{\beta y}{c^2} - \left(\frac{\omega}{c}\right)^2 \sin^2 \alpha \right\} A = 0. \quad (7.3)$$

Having changed the variables

$$\frac{\beta y}{c^2} - \left(\frac{\omega}{c}\right)^2 \sin^2 \alpha = \eta, \quad (7.4)$$

we obtain the classical Airy equation [69]:

$$\frac{\partial^2 A}{\partial \eta^2} - \left(\frac{c^2}{\beta}\right)^2 A \eta = 0. \quad (7.5)$$

The Airy function is the solution to this equation (see Fig. 16):

$$A(y) = Ai(\eta).$$

Thus, we found an exact analytical description of wave propagation in an inhomogeneous linear layer. We now consider the space-time ray propagation in the same medium under the same initial conditions.

At the  $y = 0$  boundary, the group velocity  $\mathbf{V}_g$  consists of the  $V_{0x} = c \cos \alpha$  and  $V_{0y} = c \sin \alpha$  components.

The longitudinal component of the group velocity vector  $V_x$  remains unchanged because the medium parameters are independent of the longitudinal coordinate  $x$ , and the transverse component  $V_y$  for standard STRO changes as

$$V_y(t) = V_{0y} - \int_0^t \frac{1}{2} \frac{c^2}{\omega^2} \beta \, d\tau = c \sin \alpha - \frac{1}{2} \frac{c^2}{\omega^2} \beta t. \quad (7.6)$$

The  $y(t)$  coordinate of a space-time ray varies in time  $t$  as

$$y(t) = (c \sin \alpha)t - \frac{1}{4} \frac{c^2}{\omega^2} \beta t^2. \quad (7.7)$$

The ray turning point, corresponding to the maximum value of the  $y = y_{\max}$  coordinate, is determined from the zero velocity condition  $V_y(t_{\max}) = 0$ .

The time  $t_{\max}$ , corresponding to the passage of the  $y_{\max}$  coordinate, results from (7.6):

$$t_{\max} = 2 \frac{\omega^2 \sin \alpha}{c \beta}. \quad (7.8)$$

By substituting (7.8) into (7.7), we obtain

$$y_{\max} = \frac{\omega^2 \sin^2 \alpha}{\beta}.$$

When passing from the  $y$  coordinate to the  $\eta$  coordinate (7.4), we obtain

$$\eta_{\max} = 0. \quad (7.9)$$

Thus, we can state that, within the scope of standard STRO, a ray turns in a linear layer  $\omega_L^2$  at altitudes corresponding to the zero argument of the Airy function independently of frequency  $\omega$  and wave angle of incidence  $\alpha$ .

The situation is substantially different for the modified STRO version. An additional correcting term

$$\frac{c^2 \omega_L^3}{\omega^4} \nabla_{\perp} \omega_L$$

contains a transverse (in the ray coordinates) partial derivative of the medium parameter  $\omega_L$ . If a ray is incident on a medium,  $\nabla_{\perp} \omega_L = 0$ ; therefore, rays in the standard and modified versions are identical. In both cases reflection occurs at a point with coordinate  $\eta = 0$ .

With decreasing angle  $\alpha$ , a modified ray will gradually penetrate into the region where argument  $\eta$  is positive.

For small angles of wave incidence, a difference in the ray behavior will be maximal. In this case, the group velocity vertical component for the modified STRO version can be described by the following equations:

$$\frac{dV_y}{dt} = -\frac{1}{2} \frac{c^2 \beta}{\omega^2} + \frac{1}{2} \frac{c^2 \beta^2 y}{\omega^4}; \quad (7.10)$$

$$V_y(t) = c \sin \alpha - \frac{1}{2} \frac{c^2 \beta}{\omega^2} t + \frac{1}{2} \frac{c^2 \beta^2 y}{\omega^4} t; \quad (7.11)$$

$$y(t) = (c \sin \alpha) t - \frac{1}{4} \frac{c^2 \beta}{\omega^2} t^2 + \frac{1}{4} \frac{c^2 \beta^2 y}{\omega^4} t^2. \quad (7.12)$$

Joint solution of Eqs. (7.11) and (7.12) gives the values of the maximum height and the time when a space-time ray propagates at this height:

$$y_{\max} = \frac{\omega^2}{\beta}; \quad (7.13)$$

$$t_{\max} = \frac{2\omega^2}{c\beta \sin \alpha}. \quad (7.14)$$

Passing to coordinate  $\eta$ , we obtain

$$\eta_{\max} = \frac{\omega^2}{c^2} (1 - \sin^2 \alpha). \quad (7.15)$$

In contrast to the standard STRO version, the turning point depends here on frequency  $\omega$  and angle of incidence  $\alpha$ .

### § 3. Discussion of results

First, we recall a difference between two wave propagation effects: refraction and diffraction. For quasi-monochromatic waves, refraction is an insignificant change in the wave propagation direction relative to the wavelength and a slow change relative to fast field oscillations. In terms of symbols, this is expressed in (4.1). At the same time, (4.1) is the condition of STRO applicability because this asymptotic form was specially created in order to describe refractive effects. The remaining phenomena, which cannot be defined by (4.1), are diffractive effects.

For the broadband and ultra-broadband wave packets, for which the concept of a wavelength or the division of oscillations into fast and slow phenomena have no sense, by refraction we mean a phenomenon, the spatial and temporal scales of which considerably exceed its initial dimensions along corresponding coordinates. Otherwise, wave propagation phenomena belong to diffraction.

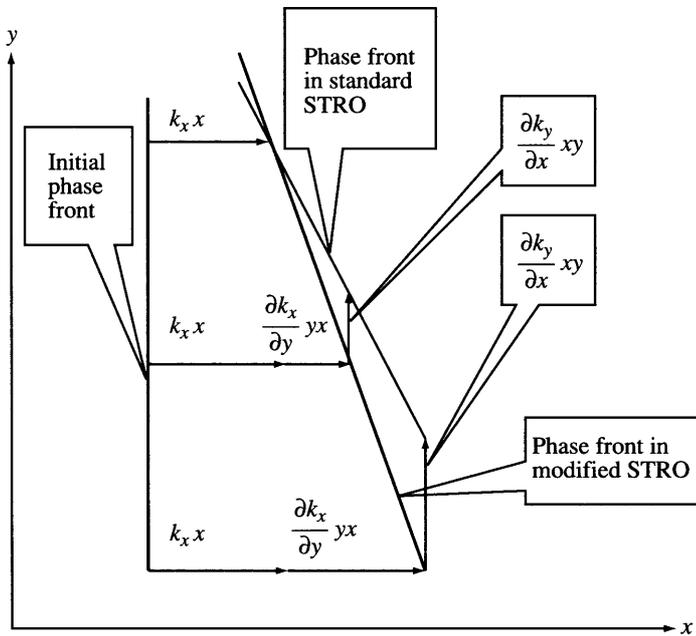
The above exact solution to the Klein–Gordon wave equation in the form of the Airy function is true for an arbitrary angle of wave incidence

$\alpha$  on an inhomogeneous layer. In the case of vertical incidence ( $\alpha = \pi/2$ ), a monotonically decreasing branch of the Airy function for  $\eta > 0$  is commensurable with a wavelength and describes a diffraction field weakening, depending on the vertical coordinate. In this case both STRO versions give an identical result, the physical sense of which is doubtless: a space-time ray, describing refractive effects, turns at a point with coordinate  $\eta = 0$ .

However, for small angles of incidence  $\alpha$ , the spatial scale of a decreasing branch of the Airy function can correspond to an arbitrarily large number of wavelengths at  $\alpha \rightarrow 0$ . In this case field weakening is smooth for  $\pi > 0$  at a wavelength and belongs to refractive phenomena.

The modified STRO version reflects the physics of this process: a space-time ray begins to penetrate into the  $\pi > 0$  area with decreasing angle of wave incidence, whereas the standard version does not respond to a change in angle  $\alpha$ . This example demonstrates that the systematic error exists in the standard STRO version, when refractive wave processes are described.

As was determined in Chapter 4, the systematic error of standard STRO is caused by a plane field model, according to which the phase function should be vortex-free, and this requires the fulfillment of the condition  $\partial k_x / \partial y = \partial k_y / \partial x$ . The modified field model compensates the  $\partial k_y / \partial x$  derivative because the amplitude function second derivative is introduced



**Fig. 36.** Spatial position of the phase front for a quasi-monochromatic non-modulated wave for the standard and modified STRO versions

into the eikonal equation, and this makes it possible to eliminate the systematic error of the standard model. Graphically the cause of the appearance of this error is explained in Fig. 36. The spatial position of the phase fronts indicates that standard STRO gives too large refraction values.

We now estimate the maximum possible correction for the Earth's ionosphere, which is taken into account in modified STRO and is absent in standard RO.

A refraction value is numerically characterized by the derivative of the group velocity vector  $d\mathbf{V}_g/dt$ .

Assume, that a quasi-monochromatic wave with frequency  $\omega$  propagates along a vertically inhomogeneous ionosphere. The transverse derivative of the group velocity for modified STRO can be written as

$$\frac{dV_y}{dt} = -\frac{c^2 \omega_L}{\omega^2} \left(1 - \frac{\omega_L^2}{\omega^2}\right) \nabla_{\perp} \omega_L. \quad (7.16)$$

Formally, the correction factor  $(1 - \omega_L^2/\omega^2)$  in (7.16) can be arbitrarily small at  $\omega \rightarrow \omega_L$ . However, its real minimal value depends on the RO applicability conditions (4.1).

The characteristic vertical scale of the ionosphere is  $L_p = 100$  km; therefore, RO can be applied in the ionosphere up to a wavelength of  $\lambda = 10$  km. Having selected this wavelength as a boundary value, we determine that the minimal possible value of the correction factor is  $(1 - \omega_L^2/\omega^2) = 0.04$ .

In other words, the systematic error of standard STRO, as compared to a more accurate modified version, can reach 96% for the group velocity derivative in the Earth's ionosphere.

#### § 4. Radiowave emission from the Earth's surface

In this and the next paragraphs, we will present the results of ray computations for standard and modified STRO. For this purpose, we will use formulas (4.13) and (7.1), respectively.

We consider the propagation of HF radio waves through the ionosphere ignoring sphericity of the ionosphere and the Earth's surface, which is considered to be quite admissible on a one-hop path. We select the ionospheric model in the form of one parabolic layer [34], the plasma frequency of which  $f_L = \omega_L/2\pi$  depends only on the vertical coordinate  $y$  and is shown in Fig. 37.

The projections of the space-time rays on the  $(x, y)$  plane in the case when a transmitter is at a point with coordinates  $x = 0, y = 0$  are shown in Fig. 38 and 39 for two versions of ray tracing. In both cases the wave frequency is  $f = \omega/\pi = 30$  MHz, and the angle of ray incidence on the ionosphere  $\alpha$  changed from 0.2 to 0.65 rad. It is almost impossible to detect a difference between the standard (Fig. 38) and modified (Fig. 39) STRO versions. This difference is observed only when a difference in the

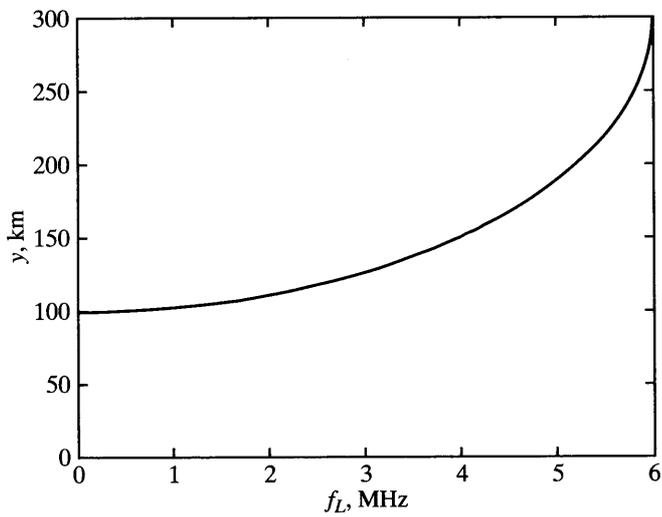


Fig. 37. The dependence of ionospheric plasma frequency on altitude

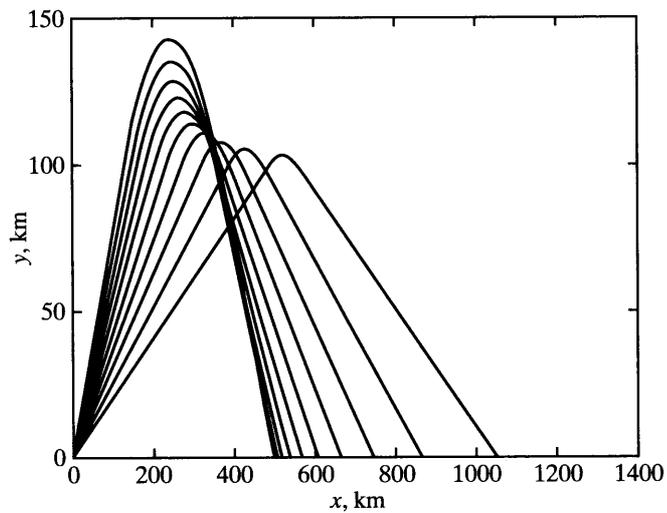
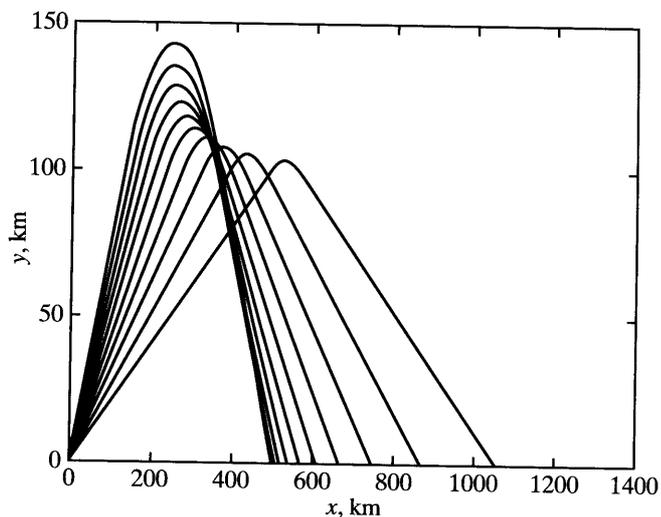


Fig. 38. Projections of the space-time rays on the spatial coordinates. Standard STRO version

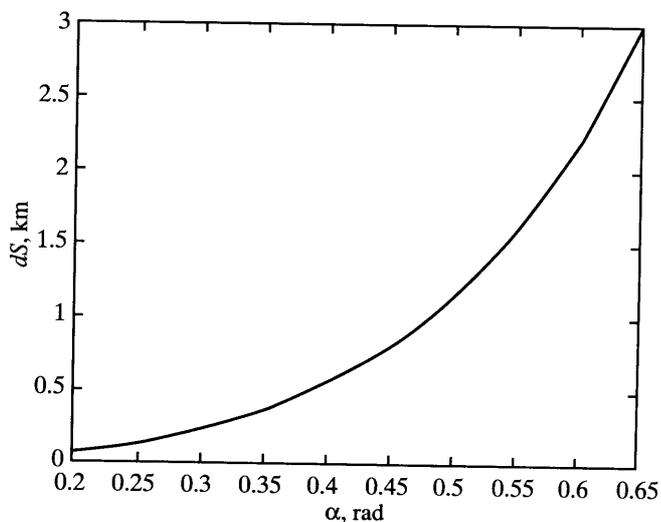
ray length  $dS$  is calculated using the formula

$$S = \int_0^T \mathbf{V}_g dt. \quad (7.17)$$

Here  $T$  is the time of ray passage along the path to the Earth's surface  $x = 0$ . A difference in the ray length is maximal  $dS = 3$  km (Fig. 40) at



**Fig. 39.** Projections of the space-time rays on the spatial coordinates. Modified STRO version



**Fig. 40.** A difference in the rays length between modified and standard STRO as a function of the angle of incidence on ionosphere

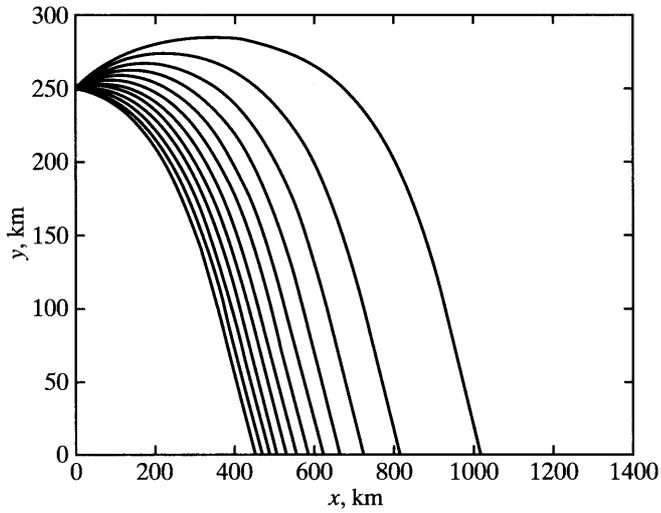
a maximal angle of incidence  $\alpha = 0.65$  rad. It is clear that this value is almost insignificant on a path longer than 1000 km, if we should not calculate wave phase characteristics when solving the problem.

This is physically explained by the fact that, according to this problem statement, a wave appears in such ionospheric regions and at such angles that the systematic error of standard STRO cannot be significant.

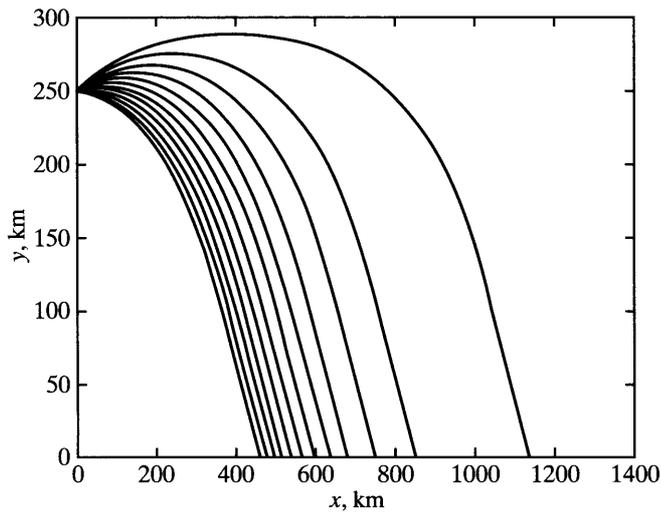
## § 5. Radiowave emission from satellite

Within the scope of the same ionospheric model (Fig. 37), we remove the radiowave source of the same frequency ( $f = 30$  MHz) from the Earth's surface  $y = 0$  to the height  $y = 250$  km, where satellites can fly.

The results of ray calculations at angles of  $\alpha = -0.025-0.25$  rad for standard and modified STRO are shown in Fig. 41 and 42, respectively.



**Fig. 41.** Projections of space-time rays on the spatial coordinates. Standard STRO version



**Fig. 42.** Projections of space-time rays on spatial coordinates. Modified STRO version

It is clear that the points of upper ray incidence on the Earth's surface are located at a distance of approximately 120 km from one another, which accounts for 10% of the path length. Such an error is inadmissible for many practical problems.

In contrast to the problem considered in the previous paragraph, emission from a satellite can lead to considerable errors when  $\omega \rightarrow \omega_L$ , and these errors can reach critical values of 96% (see §3 of this chapter).

### § 6. Wave propagation through the E–F “valley” in the ionosphere

Let us consider a more complex and real ionospheric model with a decreased plasma frequency between the E and F layers, the so-called “valley”. This model is shown in Fig. 43. The spatial ray trajectories for standard and modified STRO are shown in Fig. 44 and 45, respectively. The same initial data were used in the computations: the frequency is 30 MHz, and the angle of ray incidence on the ionosphere  $\alpha$  changed from 0.45 to 0.65 rad.

It is clear that central rays behave absolutely differently in the cases of standard and modified STRO. Therefore, we can conclude that the systematic error of standard STRO is inadmissibly large in the problems

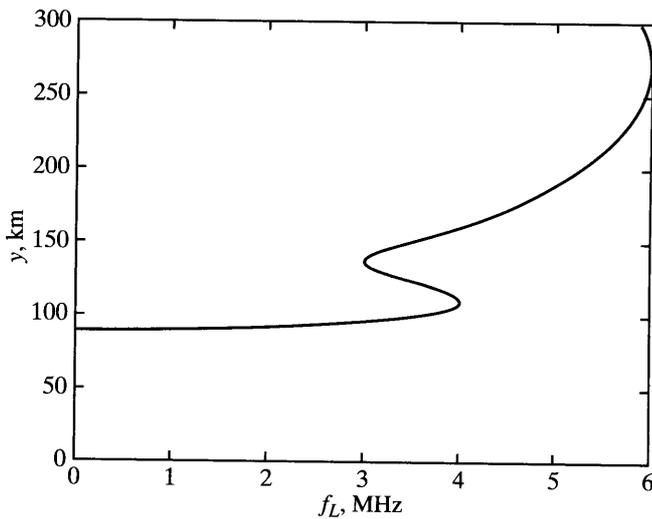
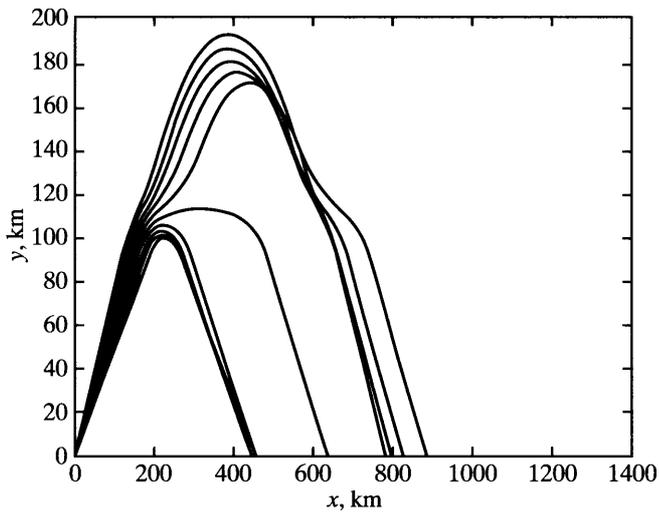
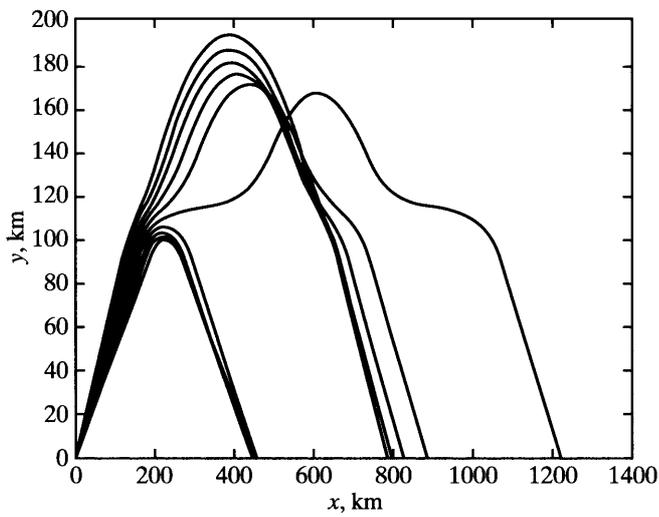


Fig. 43. Dependence of the ionospheric plasma frequency of altitude



**Fig. 44.** Projections of space-time rays on the spatial coordinates. Standard STRO version



**Fig. 45.** Projections of space-time rays on the spatial coordinates. Modified STRO version

related to wave energy canalization and propagation in the interlayer channel. At the same time, the remaining rays, which form the basis for a hoping mode, behave approximately identically in both computation versions.

## § 7. Discussion of results

The main aim of the performed numerical ray tracing is to demonstrate that many situations arise during wave propagation in the ionosphere, when the systematic error of the standard STRO version can lead to wrong results.

Such a situation can arise, e.g., when radiowaves are emitted (or received) from a satellite, when the ionospheric waveguide canalization between the E and F layers is calculated, and in almost all problems with wave phase characteristics of importance.

Nevertheless, we should note that the error of standard STRO almost does not affect the result in many problems associated mainly with oblique-incidence radio propagation. In this case radiowaves emitted from the Earth's surface do not fall in the ionospheric regions, where the error of standard STRO is large, because of the problem geometry.

However, this does not indicate that we should avoid the modified STRO version when solving these problems because a more exact method is always better than a less exact one, with other conditions being equal. This is especially important if we take into account that the modified formulas for non-modulated waves differ insignificantly from the standard formulas and almost do not affect the computation speed and necessary memory capacity.